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Self-formed straight rivers with equilibrium banks and mobile bed. Part 1. The sand–silt river

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Rivers and canals with perimeters composed of non-cohesive sand and silt have self-formed active beds and banks. They thus provide a most interesting fluid flow problem, for which one must determine the container as well as the flow. If bed load alone occurs across the perimeter of a wide channel, gravity will pull particles down the lateral slope of the banks; bank erosion is accomplished and the channel widens. In order to maintain equilibrium, this export of material from the banks must be countered by an import of sediment from the channel centre.

The mechanism postulated for this import is lateral diffusion of suspended sediment, which overloads the flow near the banks and causes deposition. The model is formulated analytically with the aid of a series of approximate but reasonable assumptions. Singular perturbation techniques are used to define the channel geometry and obtain rational regime relations for straight channels. A comparison with data lends credence to the model.

It is hoped that a first step has been made towards a more general treatment, which would include various complicating factors that are important features of natural rivers but are not essential to the maintenance of channel width. Among these factors are meandering, sediment sorting and seepage.

1. Introduction

Alluvial rivers possess channels that are self-formed by the flow of sediment-laden fluid. A considerable body of analytical work exists concerning the flow and sedimentary forms in channels of specified geometry (e.g. Kennedy 1963; Engelund 1970). When the question as to how rivers form and maintain their channels is raised, however, the analytical literature evaporates to leave a few qualitative descriptions.

Einstein (1972) notes, 'It is much less well known and appreciated that wash [suspended] load is continuously being deposited on the banks of the active channels of alluvial rivers .... As long as the river is neither widening nor narrowing its channel, bank material is being scoured and deposited .... If a stream maintains an equilibrium width, this status must be interpreted as a statistical equality of the rate of scour and the rate of deposition.'

In this paper the above statement is formulated analytically for the simplest possible case, that of a wide, laterally symmetrical, straight channel in uniform non-cohesive suspendable sand or silt carrying a constant discharge. The bed is assumed to be covered with dunes. In most field cases the problem is complicated by such factors as cohesive material, vegetation, meandering, etc. Thus the analysis is not meant to give a pretence of generality. It rather attempts to provide a foundation for a more general model; for surely although such factors as vegetation, cohesive material and meandering affect channel geometry strongly, none is essential for the existence of a self-formed channel. Furthermore, any mechanism which can stabilize field-scale channels in the absence of these factors may be thought to be of importance in their presence as well.

According to Einstein's statement, equilibrium channels can be maintained if bank scour is balanced by bank fill. Bank erosion can occur as material is removed by either lateral bed load due to gravity or entrainment into suspension. In a straight wide channel of smooth symmetrical cross-sectional shape, secondary currents can be expected to be weak and no effective mechanism exists to move bed load up from the bed to the banks. (This point is discussed in a separate section later.) Thus bank fill must be deposited from suspension.

The gravitational mechanism for bank scour is recognized by Hirano (1973) and Smith (1974) in their pioneering studies of self-formed channels. Neither, however, includes a mechanism to balance this scour and maintain a constant width. They predict channels that widen in the downstream direction even with constant water and sediment discharges. Herein the addition of a description of bank deposition is shown to lead to equilibrium channels.

The case of gravel rivers in which neither bank nor bed material can be suspended is treated in the companion paper (Parker 1978).

2. Self-formed rivers and canals in non-cohesive suspendable material

The non-cohesive channel as postulated in the previous section has some close field analogues. Sand predominates in the bed, banks and flood plain of considerable reaches of many of the streams arising in the Nebraska Sandhills, U.S.A., such as the Elkhorn River (Brice 1974), the Niobrara River (Colby & Hembree 1955) and the Middle Loup River (Hubbell & Matejka 1959). Bridge & Jarvis (1976) have described a reach of the River South Esk, Scotland, in which cohesive material is again almost entirely absent. Except for the deepest portion, where gravel is exposed, the channel is composed of sand. The existence of low natural levees indicates the ease with which this sand is suspended.

The alignment of these reaches varies from highly meandering (Elkhorn) to straight (Middle Loup). The limiting case of an essentially straight, non-cohesive, laterally symmetrical channel with constant discharge and with dune resistance is achieved in seven of the irrigation canals studied by Simons & Albertson (1963). These channels are observed to transport sand and silt in suspension, and the bank material shows an upward fining sequence, suggesting its deposition from suspension.

3. The hypothesis

The cross-section in figure 1 is considered. The channel has width B, constant downstream slope S, centre depth D_c and a perimeter of non-cohesive sand or silt with diameter D_s. The lateral distance from the left water margin is y and the depth is $D(y)$. The bank region is assumed to be of moderate curvature; if L is the distance from the water margin to the point where $D(y) = 0{\cdot}99\,D_c$, then $(D_c/L)^2$ is assumed to be

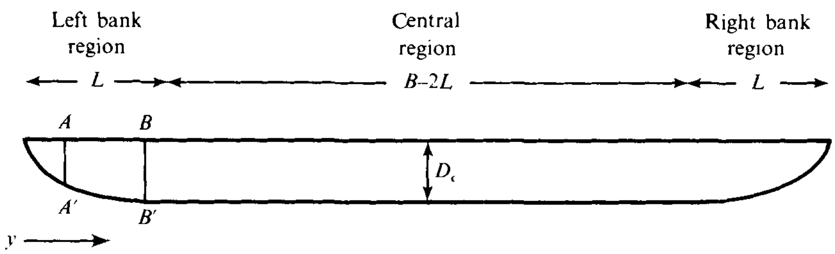

FIGURE 1. Definition diagram.

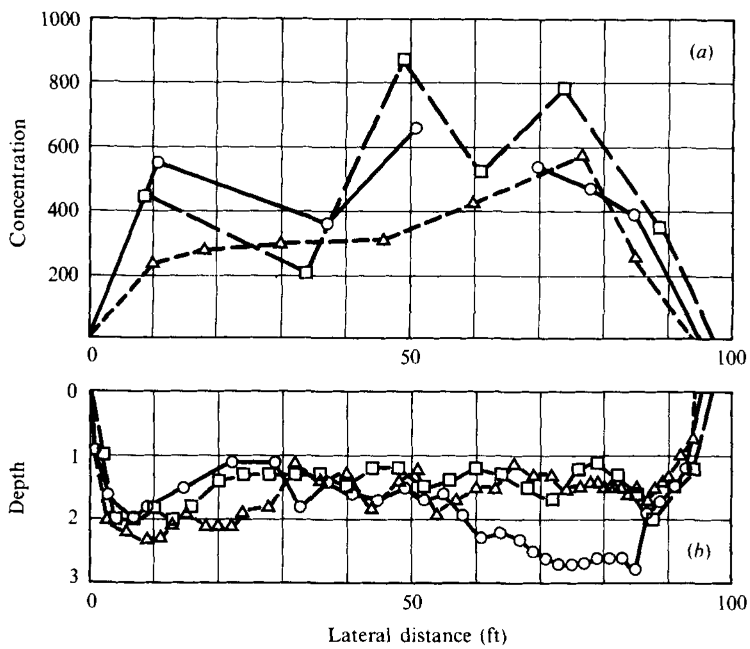

FIGURE 2. Lateral profiles of (*a*) vertically averaged suspended sediment concentration in parts per million and (*b*) depth in feet, section *B*, Middle Loup River (from Hubbell & Matejka 1959). ○—○, 26 June 1951; △—△, 16 August 1951; □—□, 6 May 1952.

small. The channel is divided into an essentially flat central region where $D \geqslant 0{\cdot}99\,D_c$ and bank regions where $D < 0{\cdot}99\,D_c$.

Of the two vertical sections AA' and BB' in figure 1, the former is the shallower. In so far as the downstream slope S is constant over the cross-section, the friction velocity $u_* \simeq (gDS)^{\frac{1}{2}}$ (g is gravitational acceleration) and thus the level of turbulence can be expected to be lower in the shallower section, implying a reduction in the ability to support suspended sediment. For example, in figure 2, the lateral distribution of suspended sediment over a fairly symmetrical cross-section of the Middle Loup River near Dunning, Nebraska exhibits a consistent tendency to decrease from the centre to the banks (Hubbell & Matejka 1959). Returning to figure 1, a lateral concentration gradient from AA' to BB' and a net diffusion of suspended sediment from the centre to the banks can be expected. The shallower banks thus become overloaded; the excess must be deposited, causing a decrease in depth. Likewise, this export from the central region leaves it underloaded; the deficiency must be obtained from the bed, causing an increase in depth.

Equilibrium is achieved by the return of the deposited bank sediment to the central region owing to a component of bed load directed down the lateral slope under the influence of gravity.

If $\mathscr{F}_L$ is the vertically integrated lateral volumetric transport of suspended sediment and q_{BL} is the lateral volumetric bed load, both per unit downstream length, a condition for equilibrium is

$$\mathscr{F}_L + q_{BL} = 0$$

for all values of y. If $\mathscr{E}$ is the volumetric rate of erosion into suspension per unit bed area and $\mathscr{D}$ is the corresponding deposition rate, then $\mathscr{D}$ must exceed $\mathscr{E}$ close to the water margin, and $\mathscr{D}$ must be less than $\mathscr{E}$ on the bank region near $y = L$ in order for $\mathscr{F}_L$ to be directed towards the banks.

4. Erosion and deposition on the bed of infinitely wide channels

A correct description of overloading and underloading is essential to the formulation of bank deposition. It can be obtained through the use of a proper bed boundary condition for suspended sediment. To this end the vertical volumetric flux of suspended sediment above a horizontally uniform channel of infinite width is considered.

Let z be height above the bed. The vertical flux F_z consists of two components: the convective flux $F_{zC} = -v_s c$, where v_s is fall velocity and c is volumetric sediment concentration, and the diffusive flux F_{zD}, which is assumed to be Fickian and of the form

$$F_{zD} = -\epsilon(z)\, \partial c/\partial z.$$

Here ϵ is a kinematic eddy diffusivity.

The bed is located at $z = a$, where $a = 0$ if a vertically constant eddy diffusivity is used and a is a few grain diameters above the bed if the Rousean eddy diffusivity is used (to avoid singular behaviour at $z = 0$). The volumetric rate at which suspended sediment is supplied to the fluid from the bed is given by the net rate of erosion $\mathscr{E} - \mathscr{D}$.

The correct bed boundary condition is that the vertical flux of suspended sediment should be continuous at the bed:

$$\lim_{z \to a} F_z(z) = [-v_s c - \epsilon \partial c/\partial z]_{z=a} = \mathscr{E} - \mathscr{D}. \qquad (1a)$$

The mechanism for bed deposition is the fall velocity. If the bed neither attracts nor repels particles that fall on it, the process is passive, and thus

$$\mathscr{D} = v_s c|_{z=a}. \qquad (2)$$

(Attraction and repulsion become important for electrically charged cohesive material.) For non-cohesive material, then, the bed boundary condition reduces to

$$-\epsilon(z)\,[\partial c/\partial z]_{z=a} = v_s E, \qquad (1b)$$

where $E = \mathscr{E}/v_s$ is a dimensionless erosion rate. This condition applies strictly to horizontally uniform conditions that may vary in time, and can be extended to conditions that vary slightly in space as well. Since (1b) is formulated in terms of the concentration gradient, it is referred to as the gradient boundary condition.

The condition (1*b*) cannot be used unless an independent prescription of E in terms of flow conditions is available. The method adopted to accomplish this is to (*a*) evaluate E as a function of flow variables for the classical case of steady, horizontally uniform flow in which F_z vanishes everywhere (zero-flux flow) and (*b*) use this evaluation in (1) applied to slightly varying conditions. For zero-flux flow, both (1) and the condition of vanishing flux F_z at the bed, viz.

$$v_s c|_{z=a} = -\epsilon[\partial c/\partial z]_{z=a}, \quad \mathscr{E} = \mathscr{D}, \qquad (3a, b)$$

must be satisfied. From (1*b*) and (3*a*), if c_a is the value of $c|_{z=a}$ observed for zero-flux flow then

$$c_a = E. \qquad (4)$$

The quantity c_a (bed concentration) can be deduced from experiments or field data approximating zero-flux flow and related to flow conditions. An example is one based on a constant eddy diffusivity $\epsilon = 0{\cdot}077 u_* D$:

$$c_a = E = 0{\cdot}0073(u_*/v_s)^3, \qquad (5)$$

which Engelund (1970) found to be approximately valid for plane or antidune beds and values of D_s ranging from 0·1 mm to 0·28 mm.

Heretofore many researchers have followed the lead of Einstein (1950) and assumed the bed boundary condition for varying flows to be of the form

$$c|_{z=a} = c_a, \qquad (6)$$

where c_a is a prescribed function of flow conditions, e.g. (5). This form is referred to as the concentration boundary condition.

Clearly the gradient and concentration boundary conditions become equivalent for zero-flux flow, which is the classical case, where (3) and (4) hold. In this case, the balance equation is $dF_z/dz = 0$, or

$$-v_s \frac{dc}{dz} = \frac{d}{dz}\epsilon\frac{dc}{dz}, \qquad (7)$$

and the water surface boundary condition is that of vanishing flux, $F_z|_{z=D} = 0$. Application of either the gradient or the concentration boundary condition leads to the same solution:

$$c = c_0(z) = E \exp\left\{-\int_a^z \frac{v_s}{\epsilon} dz\right\}, \qquad (8)$$

where $c_0(z)$ denotes the equilibrium concentration.

The two boundary conditions are not equivalent, however, for varying flow, where (3) and (4) do not hold, even though the same form for c_a and E, e.g. (5), is used. The difference is critical to the present analysis in that only the gradient boundary condition describes overloading and underloading correctly. This is illustrated with a simple example in the appendix, in which the difference in the predicted concentration profiles is large.

5. Lateral sediment transport in self-formed channels

The gradient boundary condition is now used to formulate the model of § 3. A series of assumptions and simplifications is introduced to do this. Some are severe but all are reasonable and have been used in similar contexts by previous authors, and most have some support in terms of data. This procedure, which leads to a solution as approximate as the assumptions, is dictated by the complexity of the problem, and is not without precedent. As Engelund (1970) notes concerning his own model of fluvial bedforms, '...the model makes use of some rather drastic simplifications in order to permit a reasonably convenient mathematical description'.

The lateral diffusive flux, assumed in § 3 to drive overloading (and underloading) leading to bank deposition, is given by

$$F_y = -\epsilon_y \partial c/\partial y,$$

where ϵ_y is the lateral kinematic diffusivity. The cross-sectional specification of ϵ_y and ϵ is no easy problem, particularly in that eddy diffusivities are not locally determined. The simplest justifiable procedure is to set both equal to constants corresponding loosely to averages. Engelund (1970) justifies the approximation $\epsilon = 0{\cdot}077u_* D$ in a wide channel, and Engelund (1970) and Fredsoe (1974) hold this value constant for slightly varying flows. Okoye (1970) uses flume data to construct a chart for depth-averaged ϵ_y in which $\epsilon_y/u_* D$ increases slowly with the aspect ratio B/D. A value appropriate for the Simons & Albertson (1963) canal data (later used to test the theory) is $\epsilon_y = 0{\cdot}13u_* D$. Fischer (1973) suggests that Okoye's diagram can be applied to field conditions only if the channel and thalweg do not meander, as is assumed herein. For wide channels, constant values of ϵ_y and ϵ based on the cross-sectionally averaged flow deviate little from constant values based on the flow in the central region, where essentially $u_* = u_{*c}$ and $D = D_c$. Thus the following constant values are adopted:

$$\epsilon = 0{\cdot}077u_{*c} D_c, \quad \epsilon_y = 0{\cdot}13u_{*c} D_c.$$

The steady-state equation of mass balance of suspended sediment in the channel is

$$-\epsilon_y \frac{\partial^2 c}{\partial y^2} = \epsilon \frac{\partial^2 c}{\partial z^2} + v_s \frac{\partial c}{\partial z}. \tag{9}$$

The water surface boundary condition is that of vanishing vertical flux:

$$[\epsilon \partial c/\partial z + v_s c]_{z=D} = 0. \tag{10}$$

The bed boundary condition (1) can be used directly with little error for moderate lateral bed curvature.

The problem is simplified by integrating (9) in the vertical. Integrated equations are common in the literature on the diffusion of contaminants in rivers (Fischer 1973). An approximate integration appropriate for moderate curvature with the use of the gradient boundary condition (1 a) and (10) yields

$$d\mathscr{F}_L/dy = \mathscr{E} - \mathscr{D}, \tag{11}$$

where $\mathscr{F}_L$, the vertically integrated lateral suspended flux, is approximately

$$\mathscr{F}_L = -\epsilon_y d\zeta/dy \tag{12}$$

and

$$\zeta = \int_0^D c(z)\, dz \tag{13}$$

is the vertically integrated concentration.

Relation (11) serves as the first basic relation of the analysis. Clearly if $d\mathscr{F}_L/dy \neq 0$ then according to (11) either overloading occurs, causing deposition at the rate $\mathscr{D}-\mathscr{E}$, or underloading occurs, causing bed scour at the rate $\mathscr{E}-\mathscr{D}$.

Total (suspended and bed) sediment mass balance provides the second basic relation: if $q_{BL}(y)$ is the lateral volumetric bed load per unit downstream length,

$$d(\mathscr{F}_L+q_{BL})/dy = 0. \tag{14}$$

This condition and (11) ensure that the pattern of scour and fill due to lateral bed load just balances that due to lateral diffusion, so that a steady state is maintained.

The boundary conditions on (11) and (14) are vanishing depth and lateral bed and suspended load at both water margins:

$$\left.\begin{aligned} D(0) = 0, \quad q_{BL}(0) = 0, \quad \mathscr{F}_L(0) = 0, \\ D(B) = 0, \quad q_{BL}(B) = 0, \quad \mathscr{F}_L(B) = 0. \end{aligned}\right\} \tag{15}$$

Now q_{BL}, $\mathscr{D}$ and $\mathscr{E}$ must be related to the flow. Lateral bed load is considered first. Engelund's (1974) analysis is supported by experiment in that paper and in Engelund (1975). Neglecting secondary currents, gravitational bed load down moderate lateral slopes is related to the longitudinal bed load per unit width q_B by

$$q_{BL} = \frac{q_B}{\mu}\frac{dD}{dy}. \tag{16}$$

Here μ is a dynamic Coulomb friction coefficient; Engelund (1974) deduces a value of about 0·577 from Hooke's (1974) experiments ($D_s \simeq 0{\cdot}3$ mm). Among the many choices for q_B (see, for example, White, Milli & Crabbe 1973), Chien's (1956) modified form of the Meyer-Peter & Müller (1948) relation is fairly simple:

$$\frac{q_B}{(\mathscr{R}gD_s)^{\frac{1}{2}}D_s} = 8(\tau_G^*-\tau_{cr}^*)^{1{\cdot}5}, \tag{17}$$

where $\tau_G^* = \tau_G/\rho\mathscr{R}gD_s$, τ_G is the grain bed stress, ρ_s and ρ are the sediment and fluid densities, $\mathscr{R} = \rho_s/\rho-1$, and τ_{cr}^* is the critical Shields stress, taken to be 0·047. For most natural material, $\mathscr{R} = 1{\cdot}65$; this value is used herein. Relation (17) has been used in Engelund's (1970) study of the instability of flat beds, where $\tau_G = \tau$, the total bed stress, and its approximate validity for flat beds has been reaffirmed experimentally by Luque & Van Beek (1976). The relation can be extended to beds covered with dunes provided that τ_G^* can be related to $\tau^* = \tau/\rho g\mathscr{R}D_s$. Engelund & Hansen (1967) have proposed that

$$\tau_G^*-\tau_{cr}^* = 0{\cdot}4\tau^{*2}. \tag{18}$$

This relation is part of a method for the prediction of stage-discharge relations in rivers that has been judged to be one of the more accurate at present available (Kennedy 1975). Reducing (16) with (17) and (18) gives

$$q_{BL} = 3{\cdot}51\tau^{*3}\frac{dD}{dy}(\mathscr{R}gD_s)^{\frac{1}{2}}D_s. \tag{19}$$

(In fact Engelund & Hansen chose $\tau_{cr}^* = 0{\cdot}06$; however, the discrepancy in q_{BL} caused by the difference between this value and that in (17) is usually negligible at the comparatively high values of τ^* at which appreciable suspension occurs.)

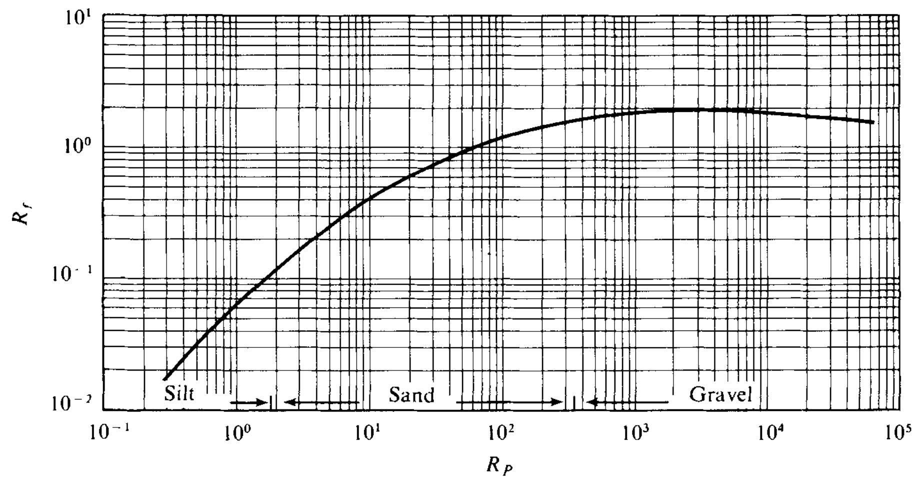

FIGURE 3. R_f *vs.* R_p calculated from the drag coefficient for spheres.

The deposition rate $\mathscr{D}$ is given exactly by (2). This form cannot be used in a vertically integrated treatment, so it is replaced with an approximate form involving ζ. From the zero-flux-flow solution (8),

$$\zeta = \frac{\epsilon}{v_s}\left[1-\exp\left(-\frac{v_s D}{\epsilon}\right)\right] c|_{z=0} \simeq \frac{\epsilon}{v_s} c|_{z=0}.$$

(The approximation was introduced by Engelund (1970) and is valid for most experimental and field fluvial sand suspensions.) Assuming, as before, that this relation holds approximately above a moderately curving bed, the approximate form

$$\mathscr{D} = (v_s^2/\epsilon)\,\zeta \tag{20}$$

of (2) is obtained and is used herein.

It is proposed that the erosion rate be evaluated from Engelund's approximate form (5), in conjunction with the condition that suspension vanishes and E is arbitrarily set equal to zero when

$$u_*/v_s < 0{\cdot}8 \tag{21}$$

(Engelund & Fredsoe 1976). This relation is more conveniently expressed in terms of Shields' stress τ^*, where $u_*/v_s = \tau^{*\frac{1}{2}}/R_f$ and $R_f = v_s/(\mathscr{R}gD_s)^{\frac{1}{2}}$ can be related to a particle Reynolds number $R_p = [(\mathscr{R}gD_s)^{\frac{1}{2}}D_s]/\nu$ (ν is the fluid kinematic viscosity) through the use of the drag curve for spheres, as shown in figure 3. Thus

$$E = 0{\cdot}0073\tau^{*1{\cdot}5}R_f^{-3}.$$

Two minor modifications are now made for convenience. This relation has been verified only for plane or antidune beds, for which $\tau^* \simeq \tau_G^*$. Under such conditions τ_G^* is so much larger than τ_{cr}^* that τ_G^* can be accurately replaced by $\tau_G^* - \tau_{cr}^*$. Furthermore, the range of particle sizes ($0{\cdot}1 \sim 0{\cdot}28$ mm) used to determine the relation is too narrow to determine accurately the dependence of E on R_f, accordingly R_f is replaced by its value for $D_s = 0{\cdot}2$ mm at 20 °C, $R_f = 0{\cdot}43$. Thus

$$E = 0{\cdot}092(\tau_G^* - \tau_{cr}^*)^{1{\cdot}5}.$$

This relation should be reasonably accurate if not applied to particle sizes much different from those used in its determination.

This form is still strictly valid only for nearly plane beds. Here it is assumed to be valid for dune beds as well and is reduced with (18), yielding

$$E = 0{\cdot}0233\tau^{*3}. \tag{22}$$

This relation is adopted herein. It should be noted that the extension of plane-bed relations pertaining to suspension to dune beds by the simple replacement of τ^* with τ^*_G is not well founded (Vanoni 1975). However this procedure, introduced by Einstein (1950), is still in common use (e.g. Engelund & Fredsoe 1976) and does not seem to provide a critical misrepresentation.

The formulation is completed with an expression for longitudinal momentum balance. It can be seen in the companion paper that for a moderately curving bed the relation

$$\tau = \rho g D S \tag{23}$$

is of sufficient accuracy.

6. Solution for the equilibrium channel

Equations (11) and (14) are now reduced and solved. The parameters $\mathscr{D}_c$ and $\mathscr{E}_c$ are the values of $\mathscr{D}$ and $\mathscr{E}$ at the channel centre, where $y = \frac{1}{2}B$, and $\mathscr{F}_L$ and q_{BL} are scaled by the constants

$$\bar{\mathscr{F}}_L = \epsilon_y \zeta_c L^{-1}, \quad \bar{q}_{BL} = 3{\cdot}51\tau_c^{*3} D_c L^{-1} (\mathscr{R} g D_s)^{\frac{1}{2}} D_s,$$

where the subscript c again denotes the value at the channel centre. The dimensionless depth s, the lateral distance from the left water margin η and the integrated concentration Z are defined as $s = D/D_c$, $\eta = y/B$ and $Z = \zeta/\zeta_c$. It turns out to be more convenient to work with $G = s^4$ than s itself. With these definitions and with the aid of (19), (20), (22) and (23), relations (11) and (14) can be expressed in the form

$$\frac{\gamma}{4}\frac{d^2G}{d\eta^2} = \lambda Z - G^P, \quad \frac{K}{4}\frac{d^2G}{d\eta^2} = \frac{d^2Z}{d\eta^2}, \tag{24), (25}$$

where $P = \frac{3}{4}$ and the relevant dimensionless constants are

$$\lambda = \frac{\mathscr{D}_c}{\mathscr{E}_c}, \quad K = \frac{\bar{q}_{BL}}{\bar{\mathscr{F}}_L}, \quad \gamma = \left(\frac{L}{B}\right)^2 \frac{\bar{q}_{BL}}{L\mathscr{E}_c}, \tag{26a, b, c}$$

It is necessary to solve only for the left half of a symmetrical channel. The six boundary conditions (15) are seen to be equivalent to

$$G(0) = 0, \quad [dG/d\eta]_{\eta=0} = 0, \quad [dZ/d\eta]_{\eta=0} = 0, \tag{27a}$$

$$G(\tfrac{1}{2}) = 1, \quad Z(\tfrac{1}{2}) = 1, \quad [dG/d\eta]_{\eta=\frac{1}{2}} = 0. \tag{27b}$$

The system represented by (24), (25) and (27) is of fourth order, yet six boundary conditions are specified, i.e. the problem is overspecified. Thus two of the dimensionless constants defined in (26) cannot be free and are determined by the solution; it turns out that these are λ and K.

Even though (24) is nonlinear, (24), (25) and (27) can be solved exactly. It is much easier in terms of analysis and physical interpretation, however, to obtain an approximate solution with the use of singular perturbation techniques. It will be found that for the typically high aspect ratios $\mathscr{A} = B/D_c$ characteristic of field channels the inaccuracy in doing this is negligible.

First (25) is integrated once and reduced with the aid of the second and third boundary conditions in (27*a*), yielding

$$\tfrac{1}{4}K\,dG/d\eta = dZ/d\eta. \tag{28}$$

It is now noted that γ can be made arbitrarily small by considering channels of increasing width B. The assumption of small γ suggests asymptotic expansions in γ of the form

$$G = G_0 + \gamma^N G_1 + \ldots,$$
$$Z = Z_0 + \gamma^N Z_1 \ldots,$$

where $N > 0$, which are inserted into (24) and (28). To lowest order, dropping the subscript,

$$\lambda Z - G^P = 0, \quad \tfrac{1}{4}K\,dG/d\eta = dZ/d\eta.$$

These relations neglect the lateral redistribution term on the left-hand side of (24), which is assumed to be responsible for bank maintenance, thus they must be valid only 'far away' from the banks, i.e. in the central region of figure 1. The appropriate boundary conditions are (27*b*); the only permissible solution determines the constant λ as well as G and Z:

$$G = 1, \quad Z = 1, \quad \lambda = 1. \tag{29}$$

In fact, (29) is an exact solution to (24) and (28) and is correct to all orders in the expansion. It is the zero-flux-flow solution, the condition $\lambda = 1$ ensuring that (3*b*), the balance between erosion and deposition, is satisfied. In the terminology of singular perturbation analysis, it is the outer solution.

The inner variable appropriate for the left bank region (a boundary layer in the analytical sense) is

$$r = \eta/\gamma^{\frac{1}{2}}.$$

Equations (24) and (28) become

$$\tfrac{1}{4}d^2G/dr^2 = Z - G^P, \tag{30}$$

$$\tfrac{1}{4}K\,dG/dr = dZ/dr. \tag{31}$$

From (27), the bank boundary conditions are

$$G(0) = 0, \quad [dG/dr]_{r=0} = 0 \tag{32a}$$

and limit matching with the outer solution requires that

$$G|_{r=\infty} = 1, \quad Z|_{r=\infty} = 1. \tag{32b}$$

As before, only the lowest-order term in the inner expansion is non-vanishing, so (30)–(32) can be solved directly, determining Z, G and K as well. The solution is

$$\left(\frac{7}{8}\right)^{\frac{1}{2}} \int_0^G \frac{dG}{(G + 3G^2 - 4G^{\frac{7}{4}})^{\frac{1}{2}}} = r, \tag{33}$$

$$Z = 1 - \tfrac{6}{7}(1 - G), \quad K = \tfrac{24}{7}. \qquad (34), (35)$$

Note that since $0 \leqslant G < 1$, (34) satisfies the rational condition $Z \geqslant 0$, i.e. that the concentration should never be negative.

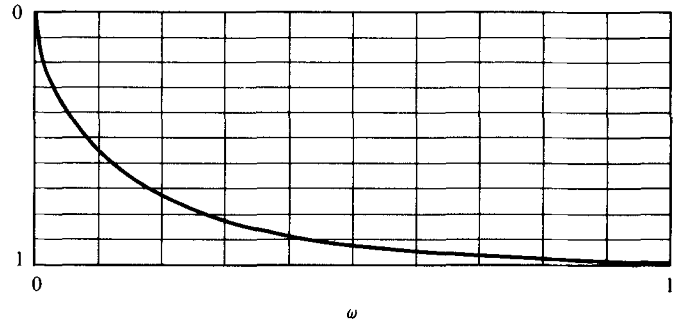

FIGURE 4. Dimensionless bank profile.

Relation (35), reduced with (26*b*), provides the most important result of the analysis: a specification of the centre depth as a function of the downstream slope, particle size and particle Reynolds number, where $R_c = D_c/D_s$ is the centre relative roughness,

$$R_c = 85{\cdot}1\, R_f^{\frac{1}{3}} S^{-\frac{1}{2}}. \tag{36}$$

The value of r at which 99 % of the centre depth is obtained, i.e. where $s = 0{\cdot}99$ ($G = 0{\cdot}961$), is called r_{99} and is found to be $r_{99} = 6{\cdot}17$. It then can be shown that

$$L/D_c = 1{\cdot}14 u_{*c}/v_s, \quad \bar{q}_{BL}/L\mathscr{E}_c = 0{\cdot}0263. \tag{37}$$

A condition that must be satisfied for the existence of a central region (as defined herein) is that $2L/B < 1$; this and (37) ensure that if a central region exists at all the condition of small γ is satisfied.

A dimensionless universal lateral profile of the bank region can be obtained from (33), and is plotted as s *vs.* $\omega = r/r_{99} = y/L$ in figure 4. The average depth of the bank region is found to be $0{\cdot}84 D_c$, and the average depth of the total channel $\bar{D}$ is related to D_c by

$$\bar{D}/D_c = 1 - 0{\cdot}32 L/B.$$

The profile in figure 4 attains infinite slope at $\omega = 0$; thus in a small portion of the bank region about this point the assumptions on which this analysis is based are not valid. In fact, the solution (33) is not valid all the way to the water margin for a number of reasons, the most compelling of which is the fact that according to (21) suspension ceases where the depth becomes too shallow. In principle, for a truly non-cohesive channel, (33) must be terminated at this point and replaced with a static equilibrium appropriate for particles so small that the critical condition for bed load is synonymous with the critical condition for suspension (D_s less than about 0·14 mm). Such a static equilibrium is described in the companion paper. However, it is coincidental that most field channels have vegetation at the water margins, the roots of which allow the profile in figure 4 to be at least qualitatively fulfilled at $\omega = 0$.

7. Comparison with data

Data for the seven non-cohesive canals listed in Simons & Albertson (1963) include values of D_{85} (particle size such that 85 % of a sample is smaller), D_{50} and D_{15} for bed and side material. Doubt immediately arises in so far as the analysis is formulated

River	Section	$Q(m^3/s)$	$D_c(m)$	$B(m)$	$S \times 10^3$	$D_{15}(mm)$
Middle Loup	A	11·1	0·210	87·8	1·29	0·18
	B_1	10·7	0·483	28·6	1·33	0·20
	C_2	11·5	0·472	25·0	1·34	0·23
	E	11·1	0·324	44·2	1·19	0·20
Niobrara	$C2$	9·6	0·396	33·5	1·23	0·18
	$C6$	9·9	0·316	40·5	1·40	0·175

TABLE 1. Characteristics of cross-sections of Middle Loup and Niobrara Rivers used to test (36).

	Low	High
Q (m³/s)	4·5	11·5
D_c (m)	0·210	1·12
B (m)	9·0	87·8
$S \times 10^3$	0·190	1·40
D_{15} (mm)	0·147	0·316

TABLE 2. Range of various parameters in data used to test (36).

for uniform material. An appropriate 'effective' particle size D_e that dominates in the postulated bed–bank exchange might be among the smaller sizes. The data indicate that few particles of the sizes near D_{50} of the bed are in suspension; these sizes are also not abundant in the banks. Particles of the size D_{15} of the bed material are, however, abundant in the banks and D_{15} might be an appropriate estimate for D_e. The data provide reasonable agreement with this choice, and it has been adopted herein without further adjustment.

In order to increase the amount of data used to test the theory, information for four cross-sections of the Middle Loup River (Hubbell & Matejka 1959) and two cross-sections of the Niobrara River (Colby & Hembree 1955) was used. Other cross-sections therein were excluded owing to either a lack of data or explicit indications of locally inerodible banks. The cross-sections of the Middle Loup River are on straight reaches; those of the Niobrara are at locally straight regions of a weakly meandering reach. Both these streams flow through the Nebraska sandhills and have unusually uniform discharges maintained by groundwater accretion. The cross-sections are adjacent to sites where extensive measurements of total and suspended load have been taken. The bankfull discharge was not listed; the dominant discharge was chosen to be that which if it continued constantly for one year would carry the total load for the year. This was found to be 11·0 cumecs for the Middle Loup and 9·9 cumecs for the Niobrara. The cross-sectional surveys that were conducted at conditions closest to the adopted dominant discharges were chosen. The width of these cross-sections allows the accurate approximation of D_c with the average depth; it also suggests a small upward adjustment of ϵ_y, which has not been performed for the sake of simplicity. Cross-sectional characteristics are described in table 1.

The ranges of Q (the observed water discharge), B, S, D_c and D_{15} (the size of the bed material) for the data are given in table 2.

In figure 5, centre depths calculated from (36) are compared with observed values. In so far as the theory is approximate and the effective particle size has been chosen

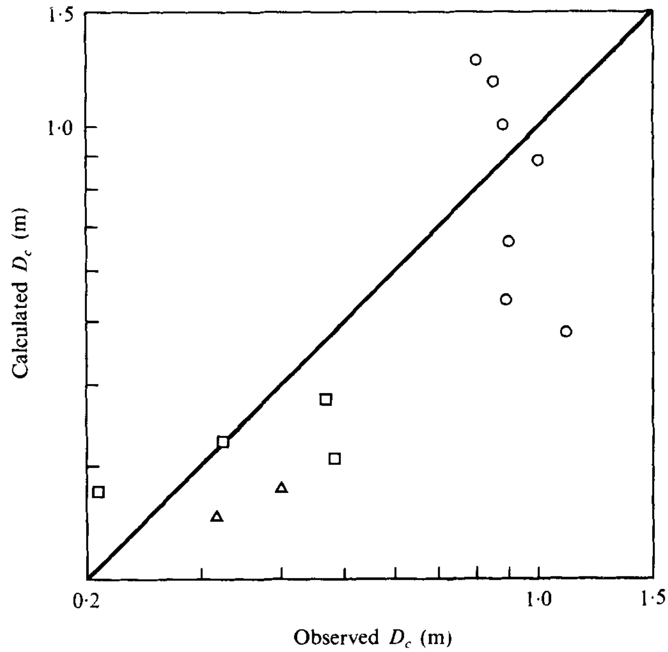

FIGURE 5. Comparison of equation (36) with data. ○, non-cohesive canals of Simons & Albertson (1963); △, Niobrara River; □, Middle Loup River.

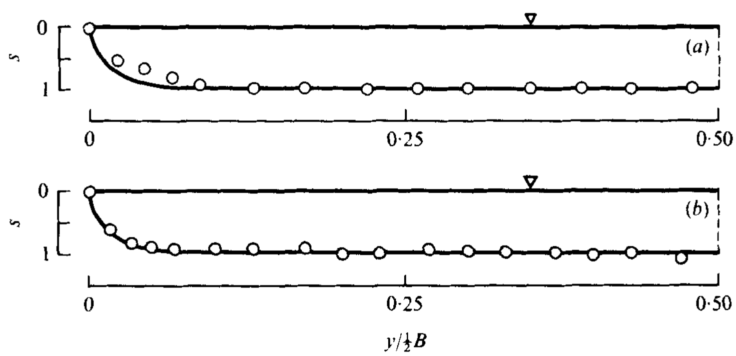

FIGURE 6. Calculated (solid line) and measured (circles) relative channel profiles for (*a*) canal no. 9, station 5 + 00 and (*b*) canal no. 23, station 4 + 00. Data are from Simons (1957). The measured values represent averages of the values for the left and right halves.

arbitrarily, one must be cautious in interpreting this figure. At the very least it indicates that the mechanism postulated for channel maintenance is a reasonable one that can predict grossly correct depths.

Calculated and observed channel profiles for two of the non-cohesive canals are compared in figure 6. The value of L was calculated from observed data and a value of D_c obtained from (36). The fits are fairly typical of the canal data.

8. Towards rational regime relations

The analysis can be used to obtain rational regime relations. Although this is not done explicitly herein, the method is indicated below. A resistance relation for the vertically averaged flow velocity V such as that due to Engelund (1970),

$$V/u_{*G} = 6 + 2{\cdot}5 \ln (R_G/2{\cdot}5),$$

where the subscript G refers to the grain resistance, and relations for the bed and suspended load allow the prescription of the water discharge per unit width $\not{q}$ and the total downstream volumetric sediment discharge per unit width q at every point y on a cross-section. The total water discharge Q and sediment discharge Q_s are given by

$$Q = \not{q}_c B \left[1 - \frac{2L}{B} \int_0^1 \left(1 - \frac{\not{q}}{\not{q}_c}\right) d\omega\right], \tag{38}$$

$$Q_s = q_c B \left[1 - \frac{2L}{B} \int_0^1 \left(1 - \frac{q}{q_c}\right) d\omega\right], \tag{39}$$

where the subscript c denotes centre conditions. Relations (36), (38) and (39) form a set of three regime relations. If ν and the characteristics of the bed material are known and any two of the parameters D_c, B, Q, Q_s and S are prescribed as independent variables [except the pair D_c and S, which is constrained by (36)], the other three can be calculated.

9. An observation concerning secondary currents; other modifications

Secondary currents in meandering channels, or meandering flows in straight channels, are known to play an important role in determining the cross-sectional channel shape (e.g. Engelund 1974; Kikkawa, Ikeda & Kitagawa 1976). Some researchers have suggested that the weaker secondary currents in straight, laterally symmetrical channels play an important role in bank stabilization (e.g. Wilson 1973). The hypothesis is that lateral cells that develop near banks have bottom flows directed bankwards, inducing a stabilizing force which opposes the erosive effect of gravity. Brundrett & Baines (1964) present results for a square duct for which the velocities of the secondary flow are not in excess of 2 % of the axial centre-line velocity. Shen & Komura (1968) summarize the results of three studies in straight rectangular ducts for which the velocities of the secondary flow do not exceed 2·5 % of the axial centre-line velocity. It seems reasonable to assume that such currents should be much weaker in straight symmetrical self-formed channels which have no sharp corners on the bed.

The stabilizing effect can be estimated as follows. According to Engelund (1974), in the absence of secondary currents the ratio of the downward lateral bed force F_g of gravity on moderate slopes to the downstream bed force F_x due to the primary flow is

$$\frac{F_g}{F_x} \simeq \frac{1}{\mu}\frac{dD}{dy},$$

where μ can be assumed to take the value given in § 5. Engelund & Skovgaard (1973) indicate that the ratio of the lateral bed force F_s induced by lateral currents to F_x obeys the approximate relation

$$F_s/F_x \simeq (u_s/u)^2.$$

Here u_s is lateral flow velocity and u is the downstream flow velocity, both measured near the bottom of the turbulent core. Even for values of u_s/u as high as 0·15, on a slope as low as $10°$, F_s/F_g is estimated to be about 0·07. This suggests the neglect of secondary currents as a stabilizing agent, as has been done herein. On the other hand, it is possible that they might act to convey suspended sediment towards the banks, thus augmenting deposition.

Of course the problem has by no means been resolved. A proper inclusion of secondary currents is only one of the many modifications that could be made. Some others are a two-dimensional approach rather than the depth-integrated approach used herein, allowance for laterally varying bed material, the use of more accurate relations for suspended and bed sediment, and detailed specification of ϵ_y and ϵ on a cross-section. A fairly sophisticated model would probably require numerical solution.

It should be noted that the straight equilibrium channel described herein cannot usually be stable with respect to meandering tendencies. However, meandering in itself does not seem to provide a mechanism for choosing a preferred width for a channel migrating in a non-cohesive flood plain. A crude model of meandering non-cohesive channels might be constructed as follows: the mechanism described herein dominates width maintenance and the flow in meander bends provides a mechanism for additional alternate erosion and deposition leading to channel migration.

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Appendix

It is assumed that zero-flux flow, with solution (8), has developed in a channel. With no change in flow conditions, the channel is now subjected to a uniform constant rain of particles, identical to those in the flow, from above the water surface. Thus overloaded conditions are imposed by a constant downward volumetric flux I across the water surface. For sufficiently small I, the bed height increases only slowly owing to deposition and a time range exists in which approximately steady, uniform overloaded conditions are realized.

The concentration is written as $c(z) = c_0(z) + e(z)$, where $c_0(z)$ is given by (8) and $e(z)$, the overload concentration, satisfies (7),

$$-v_s\frac{de}{dz} = \frac{d}{dz}\epsilon\frac{de}{dz},$$

and the water surface boundary condition

$$[\epsilon de/dz + v_s e]_{z=D} = I.$$

If the concentration boundary condition is used to complete the formulation, then $c(a)$ is a function of flow variables alone, unaffected by overloading; thus

$$c(a) = c_a + e(a) = c_a,$$

or $e(a) = 0$. This yields the solution

$$e(z) = \frac{I}{v_s}\left\{1 - \exp\left[-\int_a^z \frac{v_s}{\epsilon}dz\right]\right\}.$$

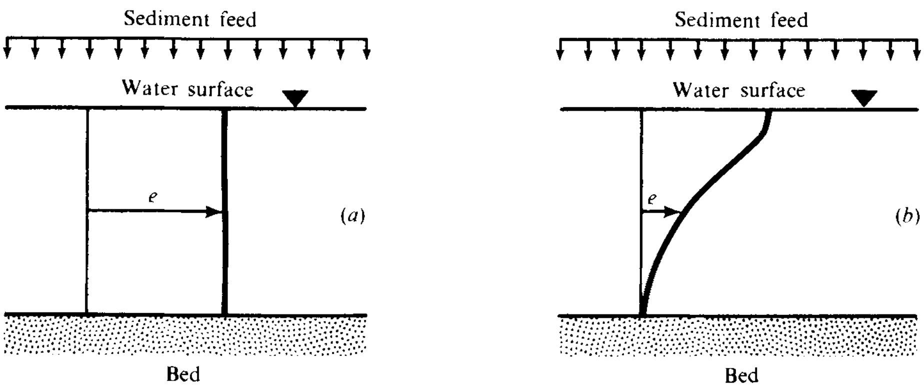

FIGURE 7. Solution for steady uniform overloading obtained with (a) the gradient boundary condition and (b) the concentration boundary condition.

The gradient boundary condition requires that the rate of erosion be unaffected by overloading, leading to $[de/dz]_{z=a} = 0$. This yields a quite different solution:

$$e(z) = I/v_s.$$

These two solutions for e are plotted in figure 7. Since the situation is that of a uniform rain of particles falling through a turbulent fluid, it is clear that the solution based on the gradient boundary condition is the correct one.

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